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The LSW Model for Domain Coarsening: Asymptotic Behavior for Conserved Total Mass

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In the classical theory of domain coarsening the particles of the coarsening phase evolve by diffusional mass transfer with a mean field. We study the long-time behavior of measure-valued solutions with compact support to this model coupled with the constraint of conserved total mass, including mean-field mass. Unlike the case of conserved volume fraction, this system has no precisely self-similar solutions, and sufficiently low supersaturation can lead to the finite-time extinction of all particles. We find a new explicit family of asymptotically self-similar solutions, and in case that the largest particle size is unbounded we establish results similar to the volume-conserved case. These include necessary criteria for asymptotic self-similarity, and sensitive dependence of long-time behavior on the distribution of largest particles in the system.

KEY WORDS: Kinetics of phase transitions; domain coarsening; conservation of mass; asymptotic behavior; self–similarity.

1. INTRODUCTION

In the classical theory of coarsening of precipitates in supersaturated solid solutions, the evolution of the particle size distribution is described by the nonlocal conservation law

$$\partial_t f + \partial_v((v^{1/3}\theta(t) - 1)\, f) = 0, \qquad v > 0, t > 0, \tag{1}$$

$$a\,\theta(t) + \int_0^\infty v f(t, v)\, dv = V. \tag{2}$$

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Here $f(t, v)$ denotes the size distribution of spherical particles of the precipitate, normalized such that $\int_{0^+}^{v} f(t, u)\, du$ is the number of remaining particles with volume less than v divided by the number at time 0. In nondimensional terms, the growth rate of a particle of volume v at time t is $v^{1/3}\theta(t) - 1$ in a uniformly supersaturated solution with mean-field supersaturation $\theta(t)$. The critical volume which decides whether the particle shrinks or grows is then $\theta(t)^{-3}$. The first equation reflects the fact that particles of finite size are neither created nor destroyed, whereas the second equation guarantees conservation of mass, taking into account matter in the diffusion field ($a > 0$ is constant).

Equivalent equations were first derived by Lifshitz and Slyozov[10] and Wagner[20] (LSW) to explain the increase of the average particle size in the aging process of a first order phase transformation. Performing an asymptotic analysis for large times, they argued that the size distribution f approaches a form which is self-similar under scaling with respect to the critical volume. They predicted that asymptotically the critical volume grows linearly in time and that the number of particles decreases proportional to t^{-1}. Their most intriguing prediction was that, independent of the initial data, almost all scaled distributions approach a particular explicit self-similar form, which in turn uniquely determines rate constants for the above power laws.

While experiments have confirmed self-similar coarsening, observed distribution functions were in general much broader and rate constants in the power laws much larger than in the LSW predictions (see e.g. ref. 8)). This deviation of the theory from experiment is explained by the mean field nature of equation (1)—the validity of this growth law is restricted to quite unrealistic situations. We do not want to go into these discussions here, but refer to Voorhees[18, 19] for the background of the problem and a review of the metallurgical and physical literature, and to ref. 12 for a mathematical justification of the growth law. The LSW equations also arise in a different context, namely as an asymptotic limit of the Becker–Döring equations.[15]

Whereas in the analysis of Lifshitz and Slyozov the total mass constraint (2) is used, Wagner argued that mass in the diffusion field can be neglected and thus the volume fraction is conserved, i.e.,

$$\int_0^\infty v f(t, v)\, dv = V. \tag{3}$$

With this constraint in place of (2), equation (1) admits exact self-similar solutions. Lifshitz and Slyozov instead argued that the supersaturation θ has to decrease to zero monotonically and with this assumption they derived the same limit behavior as Wagner.

In an earlier paper[14] the authors examined mathematically the asymptotic behavior of solutions to (1) with the volume constraint (3), for initial data with compact support. Instead of the universal behavior expected from the LSW analysis, the long-time behavior was shown to depend sensitively on the details of the initial distribution near the end of the support, that is, on the distribution of the largest particles. Among other things, convergence to the predicted LSW self-similar solution was proved impossible whenever the initial distribution is comparable to any finite power of distance to the end of the support.

A similar observation was made independently in the physical literature by Meerson and Sasorov[11] and Giron et al.[7] for data with compact support. They furthermore claim that for data with infinite support the LSW solution is approached for large times. Carr and Penrose[2] proved rigorous results for a simplified LSW equation which exhibits similar features. Velázquez[17] considered a Fokker–Planck approximation of the LSW equation, and showed that if a limit is achieved it must be the LSW solution.

In the present paper we study the long-time behavior of solutions of (1) with the total-mass constraint (2). There are two main differences between this case and that with the volume constraint (3). First, (3) implies that

$$\frac{1}{\theta(t)} = \frac{\int_0^\infty v^{1/3} f(t, v)\, dv}{\int_0^\infty f(t, v)\, dv}.$$

That is, the critical radius is equal to the mean radius in the volume-constrained system. Consequently, in this case the largest particle in the system will always grow monotonically, and without bound if there is no Dirac delta at the end of the support.

By contrast, we find that constraint (2) can imply a quite different behavior. Depending on the relation between the initial supersaturation θ_0 and the initial distribution f_0 it can happen that all particles dissolve in finite time. Also, an "intermediate" stage is possible, where the maximal particle size $\bar{v}(t)$ converges to a nonzero limit and total particle volume approaches zero while $\theta(t)$ converges to V/a. Results in this direction for a different model of crystal growth but also combined with a constraint corresponding to (2) can be found in refs. 5, 6.

The second difference between the total-mass and volume constraints is that the volume-constrained system (1) and (3) has a family of self-similar solutions while (1) and (2) does not. However, here we exhibit new exact solutions with constant scaled mean field $\bar{v}^{1/3}\theta$, which converge for large times towards self-similar profiles. In the case that the largest particle volume is unbounded in time, we can use these solutions to establish essentially all of the same results as in ref. 14 for the case of conserved volume.

The paper is organized as follows. In Section 2 we recall results we proved in ref. 13 which establish the well posedness of the initial value problem for general initial data—the initial size distribution can be an arbitrary probability measure with compact support. We also study the regularity of θ. Section 3 contains results that classify behavior for large times in the unscaled variables. There are different possibilities depending on whether the initial distribution carries a Dirac delta at the end of the support or not.

In Section 4 we study the case that the maximal particle volume $\bar{v}(t)$ increases to infinity. This largest volume is eventually monotonic in time and we can use it to rescale and describe the dynamics in terms of the scaled cumulative distribution function. We then describe exact solutions with constant scaled mean field $x = \bar{v}^{1/3}\theta$ and obtain results essentially the same as in ref. 14, giving proofs only where the arguments of ref. 14 do not immediately carry over. (It is more difficult to control the scaled mean field in the present case.) The results include the following:

(i) If the distribution function is initially comparable to a power law at the end of its support, then the scaled distribution function cannot converge to the predicted LSW form.

(ii) The possible limiting forms of the scaled distribution function are the same as those in the volume-constrained case, being classified by their exponent of vanishing $p \in [0, \infty]$. (For the LSW solution, $p = \infty$.)

(iii) For convergence to a self-similar form having $p \in [0, \infty)$, it is *necessary* that the initial distribution function be "regularly varying with exponent p" at the end of its support, as in ref. 14.

(iv) Asymptotically self-similar solutions with sufficiently small p are shape-stable to perturbations that are small in a certain restrictive sense, and for such perturbations the condition of regular variation is both necessary and sufficient for convergence to self-similar form.

The consequences of these results are further described in ref. 14. In particular, given arbitrary initial data, one can always perturb the sizes of an arbitrarily small fraction of the largest particles so that the initial distribution function satisfies the condition in (i) but is not regularly varying. It is then impossible for the perturbed solution to converge to *any* self-similar form. With respect to the natural topology for well-posedness of the initial value problem, therefore, nonconvergence occurs for a dense set of initial data.

The analysis in ref. 14 and in the present paper does not cover the case of data with infinite support. One might expect that for sufficiently nice data, e.g. data with an exponentially decreasing tail, the solution converges to the LSW–solution as claimed by Meerson et al.[11, 7] Furthermore it seems plausible, that if convergence occurs it should be towards the LSW–solution. However, it is not clear to us, what could be a necessary (and sufficient) condition for convergence towards a self–similar profile, replacing the the condition of regularly variation for data with compact support.

2. BASIC PROPERTIES OF SOLUTIONS

Since we assume $a>0$, by rescaling we can assume from now on without loss of generality that $a=1$. Throughout this paper we will deal with systems in which the particle size is initially bounded. This means that the initial size distribution is a probability measure with compact support.

In ref. 13 the authors developed a theory for well posedness of the initial value problem for this type of data. Collet and Goudon[3] studied well posedness for a family of related models for data with integrable first moment. For L^1–data Laurencot[9] generalized this result to a larger class of equations which also include the LSW model.

To discuss the results of ref. 13 we introduce some notation which we will also use later in this paper. It is convenient to work with the cumulative distribution function φ which is the fraction of initially existing particles with volume larger than v. Formally it is defined by $-d\varphi = f(t,v)\,dv$ where $f(t,v)\,dv$ is the notation for a general particle distribution which is a measure. The usual distribution function is then $F=1-\varphi$ but we will also refer to φ as the distribution function if there is no danger of confusion. The function $v\mapsto\varphi(t,v)$ is decreasing (meaning $\varphi(x_1)\geqslant\varphi(x_2)$ if $x_1\leqslant x_2$) and is taken to be right continuous. (This is as in ref. 14, where this function is erroneously described as left continuous, which was the convention taken in ref. 13. We apologize for the confusion.)

For the theory of well-posedness it turns out that it is even better to regard the volume v as a function of φ. Mathematically, this function is defined for given $\varphi(t,v)$ via

$$v(t,y) = \sup\{x \mid \varphi(t,x)>y\}\vee 0$$
$$= \begin{cases} \sup\{x \mid \varphi(t,x)>y\} & \text{for} \quad 0\leqslant y<\max\varphi(t,x), \\ 0 & \text{for} \quad y\geqslant \max\varphi(t,x), \end{cases} \tag{4}$$

where we use the notation $a\vee b=\max(a,b)$. Thus, v is decreasing and right continuous with $v(t,1)=0$. For a finite system of particles with volume ranked in decreasing order $v_0(t) \geqslant \ldots \geqslant v_{N-1}(t)$, we have $v(t, \varphi) = v_j$ for $\varphi \in [\frac{j}{N}, \frac{j+1}{N})$.

As a distance function for two size distributions with compact support we take as in ref. 13 the supremum norm between the associated volume rankings $v(t, \varphi)$ defined as above. This can be interpreted as the least "maximal volume change" required to alter one size distribution into the other. Mathematically this corresponds to the notion of the L^∞ Wasserstein distance between probability measures (cf. ref. 13).

The equation governing the evolution of $v(t, \varphi)$ is just

$$\partial_t v(t, \varphi) = v(t, \varphi)^{1/3}\, \theta(t) - 1 \tag{5}$$

for all (t, φ) such that $v(t, \varphi) > 0$. In ref. 13, we proved the following theorem on the global well-posedness of the initial value problem. Let *rcd* be the set of right-continuous decreasing functions $v_0\colon [0, 1] \to \mathbb{R}$ with $v_0(1) = 0$, with metric topology given by the supremum norm of the difference:

$$\|v_1 - v_2\| = \sup_{\varphi} |v_1(\varphi) - v_2(\varphi)|.$$

For any $T > 0$, $C([0, T], rcd)$ is the metric space of continuous $v\colon [0, T] \to rcd$, with metric given by $\sup_{[0,T]} \|v_1(t, \cdot) - v_2(t, \cdot)\|$.

Theorem 2.1 (ref. 13, Thm. 4.1). Let $v_0 \in rcd$, $V \in \mathbb{R}$. Then there exists a unique function $v \in C([0, \infty), rcd)$ such that

$$v(t, \varphi) = v_0(\varphi) + \int_0^t (v(s, \varphi)^{1/3}\, \theta(s) - 1)\, ds \tag{6}$$

whenever $v(t, \varphi) > 0$ with $\theta(t)$ determined by

$$\theta(t) + \int_0^1 v(t, \varphi)\, d\varphi = V. \tag{7}$$

Given $T > 0$, $C_0 > 0$, there exists a positive constant C such that, given two solutions as above which also satisfy $\max(V_1, V_2, v_1(0, 0), v_2(0, 0)) \leqslant C_0$ then

$$\sup_{0 \leqslant t \leqslant T} \|v_1(t, \cdot) - v_2(t, \cdot)\| \leqslant C(\|v_1(0, \cdot) - v_2(0, \cdot)\| + |V_1 - V_2|).$$

A result which is independent of the type of the conservation equation as long as $\theta(t) \geqslant 0$, is that the dynamics stretch the volume ranking vertically (that is the particle size distribution horizontally) in a smooth manner. We will see later that if $\theta_0 > 0$ then $\theta(t) > 0$ for all $t > 0$.

Proposition 2.2 (ref. 14, Prop. 2.2). Let $\theta_0 > 0$ and (θ, v) be a solution of (6) and (7) as given by Theorem 2.1. For $x > 0$, let $\mathscr{V}(t, x)$ be the solution of

$$\mathscr{V}(t, x) = x + \int_0^t (\mathscr{V}(s, x)^{1/3}\, \theta(s) - 1)\, ds$$

defined on a maximal time interval $[0, \hat{t}(x))$ where $\mathscr{V} > 0$. Then

(a) $\mathscr{V}$ is analytic in x, i.e., whenever $\mathscr{V}(t_0, x_0) > 0$, the map $x \to \mathscr{V}(t_0, x)$ is analytic near x_0.

(b) $\partial \mathscr{V} / \partial x$ is strictly increasing in time for each x.

(c) $v(t, \varphi) = \mathscr{V}(t, v_0(\varphi))$.

For the analysis to come we introduce the function $\bar{\varphi}(t)$, which denotes the end of the support of $v(t, \cdot)$, i.e.

$$\bar{\varphi}(t) = \sup \{\phi \in [0, 1] \mid v(t, \phi) > 0\}, \tag{8}$$

which is just the fraction of particles at time 0 that still exist at time t. The function $\bar{\varphi}$ is decreasing in time and right continuous, but may have jumps. We conclude from the properties of $\bar{\varphi}$ that

$$\limsup_{h \to 0} \bar{\varphi}(t+h) = \bar{\varphi}(t-0) := \lim_{h \to 0^-} \bar{\varphi}(t+h)$$

and

$$\liminf_{h \to 0} \bar{\varphi}(t+h) = \bar{\varphi}(t+0) := \lim_{h \to 0^+} \bar{\varphi}(t+h) = \bar{\varphi}(t).$$

With this notation (2) can be written as

$$\theta(t) + \int_0^{\bar{\varphi}(t)} v(t, \varphi)\, d\varphi = V \tag{9}$$

and therefore θ is continuous; indeed, it is locally Lipschitz and hence differentiable a. e. Consequently for fixed φ, $t \mapsto v(t, \varphi)$ is continuously differentiable, and satisfies $\partial_t v = v^{1/3}\theta - 1$ for all t as long as $v(t, \varphi) > 0$.

Now, with

$$\delta^h \theta(t) := \frac{\theta(t+h) - \theta(t)}{h},$$

the upper, lower, right and left derivatives of θ are defined by

$$\overline{\partial_t}\theta(t) = \limsup_{h\to 0} \delta^h\theta(t), \quad \underline{\partial_t}\theta(t) = \liminf_{h\to 0} \delta^h\theta(t), \tag{10}$$

$$\partial_t^+\theta(t) = \lim_{h\to 0^+} \delta^h\theta(t), \qquad \partial_t^-\theta(t) = \lim_{h\to 0^-} \delta^h\theta(t). \tag{11}$$

Lemma 2.3. It holds for all $t>0$ that

$$\begin{aligned}\overline{\partial_t}\theta(t) = \partial_t^-\theta(t) &= \int_0^{\bar\varphi(t-0)} (1 - v(t,\varphi)^{1/3}\,\theta(t))\,d\varphi \\ &= \int_0^{\bar\varphi(t)} (1 - v(t,\varphi)^{1/3}\,\theta(t))\,d\varphi + \bar\varphi(t-0) - \bar\varphi(t),\end{aligned} \tag{12}$$

$$\underline{\partial_t}\theta(t) = \partial_t^+\theta(t) = \int_0^{\bar\varphi(t)} (1 - v(t,\varphi)^{1/3}\,\theta(t))\,d\varphi. \tag{13}$$

Proof. With

$$\tau_\varphi := \sup\,\{t \mid v(t,\varphi) > 0\}$$

since $v(t,\varphi)=0$ for $t>\tau_\varphi$ we have

$$\partial_t v = \begin{cases} v(t,\varphi)^{1/3}\,\theta(t) - 1 & \text{for} \quad t<\tau_\varphi, \\ 0 & \text{for} \quad t>\tau_\varphi. \end{cases}$$

Using the convention $a\vee b := \max(a,b)$ and $a\wedge b := \min(a,b)$ we compute

$$\begin{aligned}\delta^h\theta(t) &= \frac{1}{h}\left\{\int_0^{\bar\varphi(t)} v(t,\varphi)\,d\varphi - \int_0^{\bar\varphi(t+h)} v(t+h,\varphi)\,d\varphi\right\} \\ &= \frac{1}{h}\left\{\int_0^{\bar\varphi(t)\vee\bar\varphi(t+h)} (v(t,\varphi) - v(t+h,\varphi))\,d\varphi\right\} \\ &= -\int_0^{\bar\varphi(t)\vee\bar\varphi(t+h)} \int_0^1 \partial_t v(t+sh,\varphi)\,ds\,d\varphi \\ &= \int_0^{\bar\varphi(t)\wedge\bar\varphi(t+h)} \int_0^1 (1 - v(t+sh,\varphi)^{1/3}\,\theta(t+sh))\,ds\,d\varphi \\ &\quad + \int_{\bar\varphi(t)\wedge\bar\varphi(t+h)}^{\bar\varphi(t)\vee\bar\varphi(t+h)} \int_0^1 -\partial_t v(t+sh,\varphi)\,ds\,d\varphi.\end{aligned}$$

Since θ and $v(\cdot, \varphi)$ are continuous and since $\bar{\varphi}(t)$ is decreasing and right continuous, $\bar{\varphi}(t) \wedge \bar{\varphi}(t+h) \to \bar{\varphi}(t)$ as $h \to 0$ and the first term converges to

$$\int_0^{\bar{\varphi}(t)} (1-v(t, \varphi)^{1/3}\, \theta(t))\, d\varphi.$$

If $h > 0$ then $\bar{\varphi}(t+h) \to \bar{\varphi}(t)$ and the second integral converges to zero since $\partial_t v$ is bounded.

If $h < 0$ then $\bar{\varphi}(t+h) \to \bar{\varphi}(t-0)$. In case $\bar{\varphi}(t-0) > \bar{\varphi}(t)$ we claim that

$$\lim_{h \to 0^-} \partial_t v(t+h, \varphi) = -1$$

for all $\varphi \in [\bar{\varphi}(t), \bar{\varphi}(t-0))$. This follows from the fact that $v(t+h, \varphi) > 0$ for all small $h < 0$, $v(t, \varphi) = 0$, and from the equation for v. ▮

So far we did not make any assumption on the datum V. Since the distribution function is always positive, $V \leqslant 0$ implies that θ is always nonpositive. Accordingly all particles shrink until they are dissolved, and no competition between particles takes place.

If $V > 0$ and $\theta(0) < 0$ then all particles will shrink in a certain time interval until they are either dissolved or until there is a time t_0 such that $\theta(t_0) > 0$. The following lemma shows that then θ will be positive for all later times.

Lemma 2.4. If $\theta(t_0) > 0$ then $\theta(t) > 0$ for all $t > t_0$.

Proof. We have seen that θ is continuous in t. Let $t_1 > t_0$ be the first time such that $\theta(t_1) = 0$. By conservation of mass (note that $\theta(t_0) > 0$ implies $V > 0$) it follows that $\bar{\varphi}(t_1) > 0$. But (13) implies

$$\underline{\partial_t}\theta(t_1) = \int_0^{\bar{\varphi}(t_1)} (1-v(t_1, \varphi)^{1/3}\, \theta(t_1))\, d\varphi = \int_0^{\bar{\varphi}(t_1)} 1\, d\varphi = \bar{\varphi}(t_1) > 0$$

which gives a contradiction. ▮

In view of the above remarks we will assume in the following without loss of generality that $V > 0$ and $\theta(0) > 0$.

3. TYPES OF LONG-TIME BEHAVIOR

The aim of this section is to classify the different types of long-time behavior of solutions to (1)–(2).

An important part will be played by the maximum particle volume at time t which we will denote by

$$\bar{v}(t) := v(t, 0) = \sup_{\varphi} v(t, \varphi).$$

The main difference between the volume constraint (3) and the total-mass constraint (2) is that in the former case one always has $\bar{v}^{1/3}\theta \geqslant 1$ and thus $\bar{v}$ is always increasing. Even though this does not hold in general here, one has the following simple but fundamental result, that for sufficiently large times the largest particle volume is monotone in time.

Lemma 3.1. If $t_0 > 0$ exists such that $\partial_t \bar{v}(t_0) = 0$, then either (i) $\operatorname{sgn} \partial_t \bar{v}(t) = \operatorname{sgn}(t - t_0)$ for t near t_0, or (ii) the solution is in equilibrium for $t \geqslant t_0$, with $v(t_0, \varphi) = \bar{v}(t_0)$ for $\varphi \in [0, \bar{\varphi}(t_0))$ and zero otherwise.

This result implies the following. Let $T_e = \infty$ if the solution never reaches equilibrium, otherwise let T_e be the first time equilibrium is reached. Lemma 3.1 implies that $\partial_t \bar{v}(t_0) = 0$ for at most one point $t_0 \in (0, T_e)$. If such a point exists then $\partial_t \bar{v} < 0$ on $(0, t_0)$ and $\partial_t \bar{v} > 0$ on (t_0, T_e). If no such t_0 exists then $\partial_t \bar{v}$ is of one sign on $(0, T_e)$ and either sign is possible.

Proof. The function $\partial_t \bar{v}$ is locally Lipschitz, and under the given hypothesis we compute that since $1 - v(t_0, 0)^{1/3}\, \theta(t_0) = 0$,

$$\underline{\partial_t}(\partial_t \bar{v})(t_0) = \bar{v}^{1/3} \underline{\partial_t} \theta = \bar{v}^{1/3} \int_0^{\bar{\varphi}(t_0)} (1 - v(t_0, \varphi)^{1/3}\, \theta(t_0))\, d\varphi \geqslant 0,$$

with equality only if equilibrium has been reached. In the case of strict inequality, the signs of $\partial_t \bar{v}(t)$ and $t - t_0$ agree for t near t_0. ∎

3.1. The Case of a Dirac Delta at the Tip

In this section we consider data f_0 which carry a Dirac delta at the end of the support. In terms of the volume ranking v this means that the data v_0 and hence also $v(t, \cdot)$ are constant on an interval $[0, \alpha)$. The results in this case show that the solution approaches some equilibrium. The number of possibilities depends on the size of V. If V is less than a critical value, then all particles are always extinguished in finite time, while if V is large enough, there are three possible long-time equilibrium states.

Proposition 3.2. Assume that for some $\alpha \in (0, 1]$ the data satisfy $v_0(0) = v_0(\varphi)$ for $0 \leqslant \varphi < \alpha$ and $v_0(0) > v_0(\varphi)$ for $\varphi > \alpha$. Then $v_\infty := \lim_{t\to\infty} \bar{v}(t)$ and $\theta_\infty := \lim_{t\to\infty} \theta(t)$ exist, and either

(i) $v_\infty = 0$ and for sufficiently large t we have $\bar{v}(t) = 0$ and $\theta(t) = V$, or

(ii) $v_\infty > 0$ is a zero of the function $F(v) := Vv^{1/3} - \alpha v^{4/3} - 1$, and $\theta_\infty = v_\infty^{-1/3}$ and $\lim_{t\to\infty} \bar{\varphi}(t) = \alpha$.

Proof. In case the solution reaches equilibrium in finite time the conclusions follow easily, so we suppose otherwise. In particular we suppose that $\bar{v}(t) > 0$ for all t. Note that $\theta(t) \leqslant V$ by (7), so if $\bar{v}(t) < 1/V^3$ at some time t then $\bar{v}$ must subsequently decrease and vanish in finite time. Hence $\bar{v}(t) \geqslant 1/V^3$ for all t.

Next, note that the assumptions on the data imply that for all times $v(t, \varphi) = \bar{v}(t)$ for $\varphi \in [0, \alpha)$ and $v(t, \varphi) < \bar{v}(t)$ for $\varphi > \alpha$. Conservation of mass now yields that

$$V > \int_0^1 v(t, \varphi)\, d\varphi \geqslant \alpha \bar{v}(t)$$

and hence $\bar{v}(t)$ is uniformly bounded. With Lemma 3.1 we get that $\lim_{t\to\infty} \bar{v}(t)$ exists.

Now,

$$v_\infty := \lim_{t\to\infty} \bar{v}(t) = \bar{v}(t_0) + \int_{t_0}^{\infty} \partial_t \bar{v}(t)\, dt < \infty.$$

From (7) and (5) it follows that $\theta(t)$ and $\partial_t v(t, \varphi)$ are uniformly bounded. Hence $v(\cdot, \varphi)$ and θ, and therefore $\partial_t \bar{v}$, are globally Lipschitz. Now it follows that $\lim_{t\to\infty} \partial_t \bar{v}(t) = 0$. This implies that $\theta_\infty := \lim_{t\to\infty} \theta(t)$ exists and $\theta_\infty = 1/v_\infty^{1/3} \in (0, V]$.

Next, since $a - b \geqslant (a^3 - b^3)/3a^2$ for $a > b > 0$, we have that

$$\begin{aligned} \partial_t(\bar{v}(t) - v(t, \varphi)) &= \theta(t)(\bar{v}(t)^{1/3} - v(t, \varphi)^{1/3}) \\ &\geqslant \frac{\theta(t)}{3\bar{v}(t)^{2/3}} (\bar{v}(t) - v(t, \varphi)) \end{aligned}$$

as long as $v(t, \varphi) > 0$. We know that $\liminf_{t>0} \theta(t) > 0$ and that $\bar{v}$ is bounded. It follows from this and the above inequality that there is a constant $c > 0$ such that

$$\bar{v}(t) - v(t, \varphi) \geqslant (\bar{v}(0) - v(0, \varphi))\, e^{ct}$$

as long as $v(t,\varphi)>0$. Since $\bar{v}$ is bounded and since $v(0,\varphi)<\bar{v}(0)$ for all $\varphi>\alpha$, it follows that for any $\varphi>\alpha$ we have $v(t,\varphi)=0$ for sufficiently large time and this proves that $\bar{\varphi}(t)\to\alpha$ as $t\to\infty$. (We also see that if $\bar{v}(0)-v(0,\alpha)>0$ then $\bar{\varphi}(t)=\alpha$ for sufficiently large t.)

With this result, (9) implies that $\theta_\infty=V-\alpha v_\infty$ and hence

$$0=\lim_{t\to\infty}\partial_t\bar{v}=v_\infty^{1/3}(V-\alpha v_\infty)-1=F(v_\infty). \quad \blacksquare$$

For a further analysis of the asymptotic behavior we concentrate on the function

$$F(v)=Vv^{1/3}-\alpha v^{4/3}-1.$$

We see that F is strictly concave in $\mathbb{R}_+$, $F(0)=-1$ and $F(v)\to-\infty$ as $v\to\infty$. Hence F has no positive zero if $V<V_*:=4\alpha^{1/4}/3^{3/4}$ and two different positive zeros $v_1<v_2$ if $V>V_*$. If $V=V_*$ then F touches the v-axis tangentially.

For fixed $\alpha\in(0,1]$ consider now the set of possible configurations of initial data containing a Dirac mass at the tip with amplitude α:

$$\mathcal{M}^\alpha:=\{v_0\in rcd: v_0(0)=v_0(\varphi) \text{ for } \varphi<\alpha,\ v_0(0)>v_0(\varphi) \text{ for } \varphi>\alpha\}.$$

Given $V\in\mathbb{R}_+$, let

$$\mathcal{M}_0^\alpha:=\{v_0\in\mathcal{M}^\alpha:\bar{v}(t)=0 \text{ for sufficiently large } t\},$$
$$\mathcal{M}_1^\alpha:=\{v_0\in\mathcal{M}^\alpha:\bar{v}(t)\to v_1 \text{ as } t\to\infty\},$$
$$\mathcal{M}_2^\alpha:=\{v_0\in\mathcal{M}^\alpha:\bar{v}(t)\to v_2 \text{ as } t\to\infty\}.$$

That is, $\mathcal{M}^\alpha$ is subdivided for given V according to the asymptotic behavior of the solution with initial data v_0.

Proposition 3.3. i) $\mathcal{M}^\alpha=\mathcal{M}_0^\alpha\cup\mathcal{M}_1^\alpha\cup\mathcal{M}_2^\alpha$.

ii) If $V<V_*$ then $\mathcal{M}^\alpha=\mathcal{M}_0^\alpha$.

iii) If $V>V_*$ then $\mathcal{M}_0^\alpha$ and $\mathcal{M}_2^\alpha$ are nonempty open subsets of $\mathcal{M}^\alpha$, and $\mathcal{M}_1^\alpha$ is nonempty and closed in $\mathcal{M}^\alpha$.

Proof. Parts i) and ii) follow directly from Proposition 3.2.

We claim $\mathcal{M}_0^\alpha$ is nonempty and open in $\mathcal{M}^\alpha$ for all $V>0$: We can always choose $\bar{v}(0)$ small enough, for example $\bar{v}(0)\leqslant 1/2V$ such that $\partial_t\bar{v}(t)\leqslant -1/2$ as long as $\bar{v}(t)>0$ and hence $\bar{v}(t)=0$ for sufficiently large t. Thus, $\mathcal{M}_0^\alpha$ is nonempty and from the continuous dependence on the data it follows that it is open in $\mathcal{M}^\alpha$.

If $V \geqslant V_*$ then $\mathscr{M}_2^\alpha$ is nonempty: We choose for example $v_0(\varphi) = 0$ for $\varphi > \alpha$. This implies $\theta_0 = V - \alpha\bar{v}(0)$ and $\theta(t) = V - \alpha\bar{v}(t)$ for all $t > 0$. If we take $\bar{v}(0) > v_1$ then $\partial_t \bar{v}(t) = F(\bar{v}(t))$ for all t and $\bar{v} \to v_2$.

If $V > V_*$ then $\mathscr{M}_2^\alpha$ is open in $\mathscr{M}^\alpha$: Let $C_1 := -F'(v_2)/2 > 0$, then for $\varepsilon > 0$ sufficiently small we have $\mp F(v_2 \pm \varepsilon) < C_1\varepsilon$. For $\varepsilon > 0$, $C_2 > 0$ define

$$N_\varepsilon = \{v \in \mathscr{M}^\alpha : |\bar{v} - v_2| < \varepsilon \text{ and } v(\varphi) < C_2\varepsilon \text{ for } \varphi \geqslant \alpha + C_2\varepsilon\}.$$

Then whenever $v(t) \in N_\varepsilon$ and ε is sufficiently small, $\partial_t v(t, \varphi) \leqslant 0$ for $\varphi \geqslant \alpha + C_2\varepsilon$ and

$$\partial_t \bar{v}(t) = \bar{v}(t)^{1/3} \left(V - \int_0^1 v(t, \varphi)\, d\varphi \right) - 1 = F(\bar{v}(t)) - \rho(t),$$

where

$$\rho(t) = \bar{v}(t)^{1/3} \left(\int_0^1 v(t, \varphi)\, d\varphi - \alpha\bar{v}(t) \right) \leqslant (v_2 + \varepsilon)^{1/3} \left((v_2 + \varepsilon) + (1 - \alpha)\right) C_2\varepsilon.$$

We can choose C_2 so that if ε is sufficiently small then $|\rho(t)| < C_1\varepsilon$. It follows that if $\bar{v} = v_2 \pm \varepsilon$ then $\partial_t \bar{v}$ and $v_2 - \bar{v}$ have the same sign. This shows that N_ε is positively invariant.

Now, if $v_0 \in \mathscr{M}_2^\alpha$ then by Proposition 3.2 we know that $v(t) \in N_\varepsilon$ for sufficiently large t. Since N_ε is positively invariant and open in $\mathscr{M}^\alpha$, from the continuous dependence on data it follows $\mathscr{M}_2^\alpha$ is open in $\mathscr{M}^\alpha$. ∎

3.2. The Case of No Dirac Delta at the Tip

Throughout this section we assume that no positive fraction of particles has maximal volume. In this case the only equilibrium state is $v \equiv 0$ with $\theta = V$. The different kinds of long-time behavior are classified by the following result. All the possibilities occur due to Proposition 3.5 below.

Proposition 3.4. Assume $\bar{v}(0) > v_0(\varphi)$ for all $\varphi > 0$. Then $\bar{\varphi}(t) \to 0$ as $t \to \infty$ and

either	$\bar{v}(t) = 0$ and $\theta(t) = V$ for sufficiently large t
or	$\bar{v}$ is decreasing on $(0, \infty)$, $\bar{v}(t) \to 1/V^3$ and $\theta(t) \to V$ as $t \to \infty$
or	$\bar{v}(t) \to \infty$ as $t \to \infty$ and $\liminf_{t \to \infty} \theta(t) = 0$.

Proof. The first alternative corresponds to the case in which the solution reaches equilibrium in finite time. We suppose this is not the case.

Then, as in the proof of Proposition 3.2, we have that $\bar{v}(t) > 1/V^3$ for all t. Lemma 3.1 implies that either $\bar{v} \to \infty$ or that $\lim_{t\to\infty} \bar{v}(t)$ exists.

Next, we claim $\liminf \theta > 0$ implies that $\bar{v}$ is bounded: Assume that $\liminf_{t\to\infty} \theta(t) > \delta > 0$. This implies that $\bar{v}(t) \leqslant 1/\delta^3$ for all t. Otherwise, since $v(t, \cdot)$ is right continuous there exists $t_0 > 0$ and $\varphi_0 > 0$ such that $v(t_0, \varphi_0)^{1/3}\,\delta - 1 > \varepsilon$ and hence $\partial_t v(t, \varphi_0) \geqslant \varepsilon > 0$ for all $t > t_0$. Thus $v(t, \varphi_0)$ is unbounded which contradicts

$$V \geqslant \int_0^1 v(t, \varphi)\, d\varphi \geqslant \varphi_0 v(t, \varphi_0).$$

We claim also, $\liminf \theta = 0$ implies $\bar{v} \to \infty$: Assume $\liminf_{t\to\infty} \theta(t) = 0$. We know from Lemma 3.1 that either $\bar{v}(t)$ is monotonically decreasing or there exists t_0 such that $\bar{v}(t)^{1/3} \geqslant 1/\theta(t)$ for all $t > t_0$. In the second case it is clear that $\liminf_{t\to\infty} \theta(t) = 0$ implies that $\bar{v}(t) \to \infty$. If $\partial_t \bar{v}(t) < 0$ for all t then also $\partial_t v(t, \varphi) < 0$ for all $\varphi > 0$ and hence $\int v(t, \varphi)\, d\varphi < V$ is decreasing. By assumption there exists a sequence $t_n \to \infty$ such that $\theta(t_n) \to 0$. But this implies $\int v(t_n, \varphi)\, d\varphi \to V$ which gives a contradiction.

Suppose now that $\bar{v}$ is bounded. Then $\liminf \theta > 0$ and as in the proof of Proposition 3.2 we have that

$$\bar{v}(t) - v(t, \varphi) \geqslant (\bar{v}(0) - v_0(\varphi))\, e^{ct}$$

for some positive c as long as $v(t, \varphi) > 0$. But this yields $\bar{\varphi}(t) \to 0$ as $t \to \infty$, hence $\int_0^1 v \to 0$ and $\theta(t) \to V$. Furthermore, by Lemma 3.1 and the argument in the proof of Proposition 3.2 it must hold $\partial_t \bar{v} \to 0$. But this implies that $\bar{v}(t) \to 1/V^3$. Note that since $\bar{v}(t) > 1/V^3$, by Lemma 3.1 $\bar{v}$ must be decreasing for all t in this case.

It remains to consider the case that $\bar{v}(t) \to \infty$ which implies $\liminf_{t\to\infty} \theta(t) = 0$. Assume that $\bar{\varphi} \to \varphi_* > 0$ as $t \to \infty$. Since θ is continuous there exists for all sufficiently small $\varepsilon > 0$ a time t_ε such that $\theta(t_\varepsilon) = \varepsilon$ and that $\theta(t) > \varepsilon$ in an interval $(t_\varepsilon - \delta, t_\varepsilon)$. But then it follows with $\int v^{1/3} d\varphi \leqslant \int (v+1)\, d\varphi \leqslant V + 1$ that

$$\underline{\partial_t} \theta(t_\varepsilon) = \int_0^{\bar{\varphi}(t_\varepsilon)} 1 - v^{1/3}(t, \varphi)\, \theta(t_\varepsilon)\, d\varphi \geqslant \varphi_* - \varepsilon(V+1) > 0$$

which gives a contradiction. ∎

Now consider

$$\mathscr{M} := \{v_0 \in rcd : v_0(0) > v_0(\varphi) \text{ for all } \varphi > 0\}$$

and for fixed $V \in \mathbb{R}_+$ denote

$$\mathscr{M}_0 := \{v_0 \in \mathscr{M} : \bar{v}(t) = 0, \theta(t) = V \text{ for sufficiently large } t\},$$
$$\mathscr{M}_1 := \{v_0 \in \mathscr{M} : \bar{v}(t) \to 1/V^3, \theta(t) \to V \text{ as } t \to \infty\},$$
$$\mathscr{M}_2 := \{v_0 \in \mathscr{M} : \bar{v}(t) \to \infty \text{ as } t \to \infty, \liminf_{t\to\infty} \theta(t) = 0\}.$$

Even though the situation here might look quite different from the case of a positive fraction of largest particles, one can view it as the formal limit as $\alpha \to 0$. Then we have $F(v) = Vv^{1/3} - 1$, $v_1 = 1/V^3$ and $v_2 = \infty$ and we obtain a result analogous to Proposition 3.3.

Proposition 3.5. i) $\mathscr{M} = \mathscr{M}_0 \cup \mathscr{M}_1 \cup \mathscr{M}_2$.

ii) $\mathscr{M}_0$ and $\mathscr{M}_2$ are nonempty open subsets of $\mathscr{M}$, and $\mathscr{M}_1$ is nonempty and closed in $\mathscr{M}$.

Proof. Part i) follows from Proposition 3.4. As in the proof of Proposition 3.3 one easily sees that $\mathscr{M}_0$ is nonempty and open in $\mathscr{M}$. If $\bar{v}(t)^{1/3} > 1/\theta(t)$ for some time t, this remains true for all later times due to Lemma 3.1 and we have $\bar{v}(t) \to \infty$ as $t \to \infty$. If we choose $\bar{v}(0)^{1/3} > 1/\theta_0$ we see that $\mathscr{M}_2$ is nonempty. By continuous dependence on the data it is clear that $\mathscr{M}_2$ is also open in $\mathscr{M}$. ∎

4. LONG-TIME BEHAVIOR II

From now on we will always consider the case that no positive fraction of particles has maximal volume and that the largest particle volume is unbounded. Contrary to the last section we will work in the following with the distribution function $\varphi(t, v)$ and not any longer with the volume ranking $v(t, \varphi)$.

Given the volume ranking v the distribution function φ can be recovered via

$$\varphi(t, y) = \sup\{x \mid v(t, x) > y\} \vee 0$$

for $y \geqslant 0$. Note that $y \mapsto \varphi(t, y)$ is right continuous with support in $[0, \bar{v}(t)]$, and $\bar{\varphi}(t) = \varphi(t, 0)$. If $\mathscr{V}$ is given as in Proposition 2.2 we have

$$\begin{aligned}\varphi(t, \mathscr{V}(t, y)) &= \sup\{x \mid \mathscr{V}(t, v_0(x)) > \mathscr{V}(t, y)\}\\ &= \sup\{x \mid v_0(x) > y\} = \varphi_0(y)\end{aligned} \tag{14}$$

if $0 < \mathscr{V}(t, y) < \bar{v}(t)$. Thus, if the data φ_0 are differentiable, φ is a solution of the advection equation

$$\partial_t \varphi + (v^{1/3}\theta(t) - 1)\, \partial_v \varphi = 0.$$

As follows from 2.5.18(3) in ref. 4, for example, the change of variables formula

$$\int_0^1 f(v(t, x))\, dx = -\int_0^\infty f(y)\, d\varphi(t, y) = \int_0^\infty \varphi(t, y)\, f'(y)\, dy \tag{15}$$

is valid for any C^1 function $f\colon (0, \infty) \to \mathbb{R}$ with compact support. Since $\varphi(t, \cdot)$ is bounded with compact support in $[0, \bar{v}(t)]$, Eq. (15) remains valid for $f(v) = v^p$ with $p > 0$. Conservation of mass may then be expressed as

$$\theta(t) + \int_0^\infty \varphi(t, v)\, dv = V.$$

The assumption $\bar{v}(t) \to \infty$ implies with Proposition 3.4 that $\liminf_{t\to\infty} \theta(t) = 0$ and $\limsup_{t\to\infty} \int_0^\infty \varphi(t, v)\, dv = V$. From Lemma 2.3 we have

$$\overline{\partial_t}\theta(t) = \partial_t^- \theta(t) = \varphi(t-0, 0) - \theta(t) \int_0^{\bar{v}(t)} \frac{\varphi(t, y)}{3y^{2/3}}\, dy, \tag{16}$$

$$\underline{\partial_t}\theta(t) = \partial_t^+ \theta(t) = \varphi(t, 0) - \theta(t) \int_0^{\bar{v}(t)} \frac{\varphi(t, y)}{3y^{2/3}}\, dy. \tag{17}$$

4.1. Rescaled Variables

From Lemma 3.1 we know that there is a time t_0 such that $\bar{v}$ is strictly increasing for all $t > t_0$. Since we are interested in the behavior for large times we assume without loss of generality that $t_0 = 0$. Now we rescale with respect to the maximal volume $\bar{v}(t)$ by introducing the new variables

$$\tau = \ln \frac{\bar{v}(t)}{\bar{v}(0)}, \qquad u = 1 - \frac{v}{\bar{v}(t)}, \qquad \psi(\tau, u) = \bar{v}(t)\, \frac{\varphi(t, v)}{V}. \tag{18}$$

For each $\tau \geqslant 0$, the function $u \mapsto \psi(\tau, u)$ is left continuous and increasing on $[0, 1]$ with $\psi(\tau, 0) = 0$. With $\psi_0(u) = \bar{v}(0)\, \varphi_0(v)/V$, Eq. (14) implies that the evolution of $\psi(\tau, u)$ is determined by the relation

$$\psi(\tau, \mathscr{U}(\tau, u)) = e^{\tau} \psi_0(u), \tag{19}$$

as long as $0 < \mathscr{U}(\tau, u) < 1$, where $\mathscr{U}(\tau, u) = 1 - \mathscr{V}(t, v)/\bar{v}(t)$ satisfies

$$\frac{\partial \mathscr{U}}{\partial \tau} = (\kappa(\tau)\, Q(\mathscr{U}) - 1)\, \mathscr{U}, \qquad \mathscr{U}(0, u) = u, \tag{20}$$

where Q is given by

$$Q(u) = 3 \left(\frac{1 - (1-u)^{1/3}}{u} \right). \tag{21}$$

The function Q satisfies $Q(0) = 1$, $Q(1) = 3$, and is given by a power series

$$Q(u) = 1 + \frac{1}{3} u + \frac{5}{27} u^2 + \cdots,$$

which converges for $|u| < 1$ and whose coefficients are all positive. (Hence $Q(u) \leqslant 1 + 2u$ for $0 \leqslant u \leqslant 1$.) The function $\kappa(\tau) > \frac{1}{3}$ is determined from

$$\frac{3\kappa(\tau)}{3\kappa(\tau) - 1} = x(\tau) := \bar{v}(t)^{1/3} \theta(t) > 1. \tag{22}$$

Conservation of mass is now expressed by

$$x(\tau)\, \frac{e^{-\tau/3}}{\gamma} + \int_0^1 \psi(\tau, u)\, du = 1 \tag{23}$$

where $\gamma = \bar{v}(0)^{1/3}\, V$.

For differentiable initial data, ψ satisfies

$$\partial_\tau \psi + (\kappa(\tau)\, Q(u) - 1)\, u \partial_u \psi = \psi \tag{24}$$

for $0 < u < 1$, $\tau > 0$, and the characteristics of this PDE are given by (20). Since $\kappa(\tau)\, Q(1) - 1 > 0$ for all τ, every point $(\tau_1, u_1) \in (0, \infty) \times (0, 1)$ lies on a unique characteristic that can be continued back to time $\tau = 0$. That is, $u_1 = \mathscr{U}(\tau_1, \tilde{u}_1)$ for some $\tilde{u}_1 \in (0, 1)$.

From (16)–(17) we compute

$$\overline{\partial_\tau} x(\tau) = \partial_\tau^- x(\tau) = \gamma e^{\tau/3} \left(\frac{e^{-\tau/3}}{3\gamma} x(\tau) + \frac{b(\tau-0)}{x(\tau)-1} - c(\tau) \right), \tag{25}$$

$$\underline{\partial_\tau} x(\tau) = \partial_\tau^+ x(\tau) = \gamma e^{\tau/3} \left(\frac{e^{-\tau/3}}{3\gamma} x(\tau) + \frac{b(\tau)}{x(\tau)-1} - c(\tau) \right), \tag{26}$$

where

$$b(\tau) = \psi(\tau, 1) - \int_0^1 \frac{\psi(\tau, u)}{3(1-u)^{2/3}}\, du, \qquad c(\tau) = \int_0^1 \frac{\psi(\tau, u)}{3(1-u)^{2/3}}\, du.$$

Note that $b(\tau) > 0$ and $c(\tau) > 0$ for all $\tau > 0$.

4.2. Solutions with Constant Mean Field

We are now interested in possible asymptotic states as $\tau \to \infty$. We first note that equation (24) coupled with (23) does not have stationary solutions. But if κ converges to a constant and ψ to a function as $\tau \to \infty$ then (for the moment formally) these limits solve the same stationary problem as (24) coupled with conservation of volume fraction. We will see that one can easily construct explicit solutions of (24) and (23) with constant κ that converge to solutions of the associated stationary problem.

Solutions of the corresponding stationary problem are increasing functions ψ_* which solve

$$\frac{d}{du} \ln \psi_*(u) = \frac{1}{u(\kappa_* Q(u) - 1)} \tag{27}$$

for $0 < u < 1$ and a constant κ_*, with the normalization

$$\int_0^1 \psi_*(u)\, du = 1. \tag{28}$$

In ref. 14, section 4, solutions of this equation are extensively studied. There it is shown in Lemma 4.1 that any function ψ_* which has finite total volume must have compact support. This is satisfied for all solutions of (27) with $\kappa_* \geqslant 1$. By integrating (27) one can easily see that solutions can be characterized by the exponent $p = 1/(\kappa_* - 1)$, which describes the vanishing behavior of ψ_* as $u \to 0$. Hence, we will write Ψ_p to denote the solution of (27)-(28) with $\kappa_* = 1 + 1/p$.

Lemma 4.1 (ref. 14, Lemma 4.2). For any $p \in (0, \infty]$ there is a solution to (27) with $\kappa_* = 1 + \frac{1}{p}$. For $p < \infty$ the profile has the form

$$\Psi_p(u) = \alpha_p(u)\, u^p$$

where α_p is decreasing and analytic on $[0, 1)$. For $p = \infty$ ($\kappa_* = 1$) we have $\Psi_\infty(u) = o(u^q)$ as $u \to 0$ for all $q > 0$.

In Chapter 4 of ref. 14 and in ref. 7 analytical expressions for the solutions are given.

In the following we denote by $\Psi_p(u)$ always the solution which is normalized such that (28) holds. This normalization implies $\alpha_p(1) \leqslant p+1 \leqslant \alpha_p(0)$, and we note that (27) yields

$$-\frac{\partial}{\partial u} \ln \alpha_p(u) = \frac{1}{u}\left(\frac{1}{\kappa_* - 1} - \frac{1}{\kappa_* Q(u) - 1}\right) = \frac{\kappa_*(Q(u)-1)}{(\kappa_* - 1)} \frac{\partial}{\partial u} \ln \Psi_p(u)$$
$$\leqslant 2u(p+1)\left(\frac{p}{u} + \frac{\partial}{\partial u} \ln \alpha_p(u)\right) \leqslant \frac{2(p+1)\,p}{1+2(p+1)\,u},$$

whence we infer that $\ln(\alpha_p(0)/\alpha_p(1)) \leqslant p \ln(2p+3)$. This yields the bounds

$$(p+1)(2p+3)^{-p}\, u^p \leqslant \Psi_p(u) \leqslant (p+1)(2p+3)^p\, u^p. \tag{29}$$

As $p \to 0$ one has $\Psi_p(u) \to 1$ for $0 < u \leqslant 1$. This corresponds to a particle distribution which concentrates into a Dirac delta at the tip of the support.

With the solutions of the stationary problem at hand we can easily construct solutions with constant κ_* to (24) and (23). Those correspond to asymptotically self-similar solutions to the equation in the original variables.

Lemma 4.2. Let $p \in (0, \infty]$ and let $\gamma > \gamma_0$ where γ_0 is sufficiently large. Then there is a nonnegative solution $\Phi_{p,\gamma}(\tau, u)$, increasing in u, to (23) and (24) for all $\tau > 0$ with constant $\kappa = 1 + \frac{1}{p}$. It has the form

$$\Phi_{p,\gamma}(\tau, u) = \Psi_p(u) - \frac{e^{-\tau/3}}{\gamma} c_p \Psi_p(u)^{4/3} \tag{30}$$

where c_p is a positive constant.

Proof. We look for a solution of (23)-(24) of the form $\Phi_{p,\gamma} = \psi_p + f(\tau)\,\tilde{\psi}$ with $\kappa \equiv \kappa_*$. If we plug this into the equation we immediately find that (κ_*, ψ_p) must be a solution of the associated stationary problem and hence we can find for all $p \in (0, \infty]$ the solution $\kappa_* = 1 + \frac{1}{p}$ and $\psi_p = \Psi_p$

as in Lemma 4.1. For the remaining terms we conclude from (23) that $f(\tau) = -e^{-\frac{\tau}{3}}$ and then it follows that $\tilde{\psi}$ must solve the equation

$$\partial_u \ln \tilde{\psi}(u) = \frac{4}{3} \frac{1}{u(\kappa_* Q(u) - 1)}$$

and must be normalized such that

$$\int_0^1 \tilde{\psi}(u)\, du = \frac{1}{\gamma} \frac{3\kappa_*}{3\kappa_* - 1} = \frac{1}{\gamma} \frac{3p+3}{2p+3}. \tag{31}$$

We immediately see that the solution is given by $\tilde{\psi} = c_p \Psi_p^{4/3}/\gamma$ where c_p is determined by the normalization.

Furthermore, it is trivial but important to note that the behavior at the tip $u = 0$ of $\Phi_{p,\gamma}$ is the same as for Ψ_p. In addition, we have to choose γ sufficiently large such that $\Phi_{p,\gamma}(\tau, \cdot)$ is increasing for all $\tau > 0$. It is easily seen that this can be done uniformly in p. ∎

4.3. First Consequences of Convergence to Self-Similar Form

Lemma 4.3. Assume that $\lim_{\tau\to\infty} \psi(\tau, u) = \psi_\infty(u)$ exists for each $u \in [0, 1]$. Then $\lim_{\tau\to\infty} 1/\kappa(\tau) = 1/\kappa_\infty$ exists, where $1 \leqslant \kappa_\infty \leqslant \infty$, and $\psi_\infty(u) = \Psi_p(u)$, where $p = 1/(\kappa_\infty - 1) \in [0, \infty]$. Moreover, as $\tau \to \infty$, $\psi(\tau, u)$ converges uniformly for u in any compact subset of $(0, 1]$.

Proof. The only step in the proof which is different from that of Lemma 5.2 in ref. 14 is to conclude from the convergence of ψ that $1/\kappa(\tau)$ converges. The rest of the proof is identical. In fact, it is always the crucial part to get control over κ.

First, we show that $\kappa_\infty := \limsup_{\tau\to\infty} \kappa(\tau) \geqslant 1$ by adapting an argument from ref. 14. Suppose $\kappa_\infty < 1$. Then since $Q(0) = 1$, there is some $u_0 > 0$ for which $\kappa_\infty Q(u) - 1 < 0$ for $0 < u \leqslant u_0$. For sufficiently large τ this means $\partial_\tau \mathscr{U} < 0$ if $\mathscr{U} \leqslant u_0$, hence there is a characteristic satisfying $0 < \mathscr{U}(\tau, u_1) < u_0$ for τ large. It follows from (19) that $\psi(\tau, u) > e^\tau \psi_0(u_1)$ for $u_0 < u \leqslant 1$ and hence $\int_0^1 \psi(\tau, u)\, du \geqslant e^\tau (1 - u_0)\, \psi_0(u_1) \to \infty$ as $\tau \to \infty$. But our hypotheses imply that $\int_0^1 \psi(\tau, u)\, du \to \int_0^1 \psi_\infty(u)\, du < \infty$. Therefore $\kappa_\infty \geqslant 1$.

As we assume that ψ converges pointwise as $\tau \to \infty$ the dominated convergence theorem and (15) imply that $\int_0^1 \psi(\tau, u)\, du \to 1$ and from mass conservation (23) we infer

$$x(\tau) \frac{e^{-\tau/3}}{\gamma} \to 0. \tag{32}$$

The convergence of ψ also gives

$$c(\tau) \to c_\infty := \int_0^1 \frac{\psi_\infty(u)}{3(1-u)^{2/3}}\, du, \qquad b(\tau) \to b_\infty := \psi_\infty(1) - c_\infty. \tag{33}$$

Let us assume first that $b_\infty = 0$, which implies $\psi_\infty(u) \equiv 1$ and hence $c_\infty = 1$. We claim that then $\lim_{\tau\to\infty} x(\tau) = 1$.

Let $\varepsilon > 0$. It follows from (25), (32) and (33) that whenever τ is sufficiently large and $x(\tau) > 1+\varepsilon$,

$$\overline{\partial_\tau} x(\tau)\, \frac{e^{-\tau/3}}{\gamma} \leqslant x(\tau)\, \frac{e^{-\tau/3}}{3\gamma} + \frac{b(\tau-0)}{\varepsilon} - c(\tau) < -\frac{1}{2}.$$

Hence $\limsup_{\tau\to\infty} x(\tau) \leqslant 1+\varepsilon$ for any $\varepsilon > 0$. Since $x(\tau) > 1$ for all τ we conclude that $x(\tau) \to 1$ as $\tau \to \infty$. That is, $\kappa(\tau) \to \infty$. This covers the case $p = 0$.

Now we consider the case $b_\infty > 0$. We define for all $x > 1$

$$F_\infty(x) = \frac{b_\infty}{x-1} - c_\infty.$$

The function F_∞ is strictly decreasing and has a zero $x_\infty = 1 + b_\infty / c_\infty > 1$. We claim that $\lim_{\tau\to\infty} x(\tau) = x_\infty$. For the proof we define

$$F(x, \tau) := x\, \frac{e^{-\tau/3}}{3\gamma} + \frac{b(\tau-0)}{x-1} - c(\tau)$$

so that

$$\overline{\partial_\tau} x(\tau) = \gamma e^{\tau/3} F(x(\tau), \tau). \tag{34}$$

Choose $\varepsilon > 0$ and let $\varepsilon_1 := -\frac{1}{2} F_\infty(x_\infty + \varepsilon) > 0$. With (32) and (33) we conclude that for some sufficiently large τ_1, whenever $\tau \geqslant \tau_1$ and $x(\tau) > x_\infty + \varepsilon$ we have

$$|F(x(\tau), \tau) - F_\infty(x(\tau))| \leqslant \varepsilon_1.$$

Hence either $F(x(\tau), \tau) \leqslant F_\infty(x(\tau)) < F_\infty(x_\infty + \varepsilon) = -2\varepsilon_1$ or

$$F(x(\tau), \tau) \leqslant \varepsilon_1 + F_\infty(x(\tau)) < \varepsilon_1 + F_\infty(x_\infty + \varepsilon) = -\tfrac{1}{2}\, \varepsilon_1.$$

From this and (34) we conclude that $\limsup_{\tau\to\infty} x(\tau) \leqslant x_\infty + \varepsilon$ for all $\varepsilon > 0$. Similarly it follows $\liminf_{\tau\to\infty} x(\tau) \geqslant x_\infty - \varepsilon$ which proves the claim.

In terms of κ we find that $\kappa_\infty = \lim_{\tau \to \infty} \kappa(\tau)$ exists. We have shown that $\kappa_\infty \geqslant 1$, and it remains only to show that $\psi_\infty(u)$ is the appropriate solution to (27). The proof is analogous to that given in ref. 14, Lemma 5.2, and thus we omit it here. ▮

4.4. A First Criterion for Nonconvergence

The following proposition based upon a comparison principle transfers directly from ref. 14 if one replaces volume conservation by the fact that

$$\limsup_{\tau \to \infty} \int_0^1 \psi(\tau, u)\, du = \frac{1}{v} \limsup_{t \to \infty} \int_0^\infty \varphi(t, v)\, dv = 1. \tag{35}$$

Proposition 4.4 (cf. ref. 14, Prop. 5.1). Let ψ be determined by (19)–(23), and suppose $0 < p < \infty$.

(a) If $\inf_{u>0} \psi_0(u)/u^p > 0$, then $\limsup_{\tau \to \infty} \kappa(\tau) \geqslant 1 + \frac{1}{p}$.

(b) If $\sup_{u>0} \psi_0(u)/u^p < \infty$, then $\liminf_{\tau \to \infty} \kappa(\tau) \leqslant 1 + \frac{1}{p}$.

From this result together with Lemma 4.3, we immediately deduce a simple criterion that forbids convergence of $\psi(\tau, u)$ to $\Psi_p(u)$ for any particular $p \in (0, \infty]$. In particular this criterion applies to the limiting form $\Psi_\infty(u)$ favored by the LSW theory.

Corollary 4.5. Suppose that for some real number q we have

$$\inf_{u>0} \frac{\psi_0(u)}{u^q} > 0.$$

Then for any p satisfying $q < p \leqslant \infty$, it is impossible that $\lim_{\tau \to \infty} \psi(\tau, u) = \Psi_p(u)$ for all $u \in [0, 1]$.

4.5. A Necessary (and Sufficient?) Condition for Convergence

Even though one might expect from the previous results that if $\psi(\tau, u) \to \Psi_p(u)$ for $0 \leqslant u \leqslant 1$ and if $p = 1/(\kappa_* - 1) < \infty$, then $\psi_0(u) \sim cu^p$ as $u \to 0$, this is not exactly true. In Theorem 5.10 in ref. 14 we give a necessary criterion for convergence which holds true here as well (Theorem 4.7 below). This criterion is related to the concept of regularly varying function as is treated in the books of Seneta[(16)] and Bingham et al.[(1)] for example.

Definition 4.6. A positive, measurable function g, defined on some interval of the form $(0, a]$, is called *regularly varying at* 0 (with exponent $p \in \mathbb{R}$) if

$$\lim_{x \to 0^+} \frac{g(\lambda x)}{\lambda^p g(x)} = 1 \qquad \text{for all} \quad \lambda > 0. \tag{36}$$

If $p = 0$, we say g is *slowly varying* at 0.

Our main result in ref. 14 gives a necessary condition for convergence in the rescaled variables to any one of the stationary solutions Ψ_p with $0 \leqslant p < \infty$. The main idea in the proof is that convergence of ψ implies convergence of κ and this in turn allows to control the characteristics. Since convergence of κ is ensured in Lemma 4.3 the arguments in the proof of Theorem 5.10 in ref. 14 apply here as well and therefore we only state the result.

Theorem 4.7 (cf. ref. 14, Th. 5.10). Assume that for some $p \in [0, \infty)$ we have

$$\lim_{\tau \to \infty} \psi(\tau, u) = \Psi_p(u) \qquad \text{for all} \quad u \in [0, 1].$$

Then ψ_0 is regularly varying at 0 with exponent p.

Our conjecture is that the necessary criterion in Theorem 4.7 is also sufficient for convergence to Ψ_p as $\tau \to \infty$. A general proof is lacking, but below we will prove sufficiency when $p > 0$ is sufficiently small and the initial data is close to Ψ_p in a certain sense. The following result reduces the general problem of sufficiency to proving the convergence of $\kappa(\tau)$.

Theorem 4.8 (cf. ref. 14, Th. 5.11). Assume that $\lim_{\tau \to \infty} \kappa(\tau) = \kappa_* \in (1, \infty]$. Then, $\lim_{\tau \to \infty} \psi(\tau, u)$ exists for all $u \in [0, 1]$ if and only if ψ_0 is regularly varying with exponent $p = 1/(\kappa_* - 1)$.

Since convergence of κ is assumed, the proof of Theorem 5.11 in ref. 14 transfers without significant changes to the present situation.

4.6. Conditional Stability for Small p

Next we show how to obtain control over $\kappa(\tau) - \kappa_*$ when $p > 0$ is small and the parameter $\gamma = \bar{v}(0)^{1/3} V$ is large enough. With this control we prove a shape-stability result for constant-mean-field solutions $\Phi_{p,\gamma}$ with respect to perturbations that are small in a sense related to the notion of regularly varying functions. For such small perturbations it is necessary and sufficient for convergence that the data are regularly varying.

First we recall a few definitions and results from ref. 14.

Definition 4.9. We say that a real-valued, measurable function h, defined on some interval $[A, \infty)$, is *locally linear at* ∞ (with *slant* $p \in \mathbb{R}$) if

$$\lim_{y \to \infty} h(y+L) - h(y) = pL \qquad \text{for all} \quad L \in \mathbb{R}. \tag{37}$$

If $p = 0$ we say h is *locally flat* at ∞.

Lemma 4.10. Suppose $h(-\ln x) = -\ln g(x)$. Then the following are equivalent.

(i) g is regularly varying at 0 with exponent p.

(ii) h is locally linear at ∞ with slant p.

We define the *oscillation* of a function h on an interval $[a, b]$ to be

$$\operatorname*{osc}_{z \in [a,b]} h(z) = \sup_{z \in [a,b]} h(z) - \inf_{z \in [a,b]} h(z) = \sup_{z_1, z_2 \in [a,b]} h(z_1) - h(z_2).$$

and the *flatness modulus* of h to be

$$\varpi(\tau, y) = \sup_{\tilde{y} \geqslant y} \operatorname*{osc}_{z \in [\tilde{y}, \tilde{y}+1]} h(\tau, z). \tag{38}$$

Note that whenever $0 \leqslant a < b < c$,

$$\operatorname*{osc}_{z \in [a,c]} h(\tau, z) \leqslant \operatorname*{osc}_{z \in [a,b]} h(\tau, z) + \operatorname*{osc}_{z \in [b,c]} h(\tau, z),$$

so for any positive integer n,

$$\operatorname*{osc}_{z \in [y, y+n]} h(\tau, z) \leqslant n\varpi(\tau, y). \tag{39}$$

Evidently, $\varpi(\tau, y)$ is decreasing in y, and we have $\varpi(\tau, y) \to 0$ as $y \to \infty$ if and only if $h(\tau, \cdot)$ is locally flat at ∞.

For the following we introduce new variables slightly different from those in ref. 14:

$$y = -\ln u, \tag{40}$$

$$h(\tau, y) = -\ln\left(\psi(\tau, e^{-y}) / \Phi_{p,\gamma}(\tau, e^{-y})\right) \tag{41}$$

and $\varpi(\tau, y)$ will from now on always denote the flatness modulus of h as defined in (41). The bound on $\varpi(\tau, 0)$ in the following theorem is a shape-stability result, since if the flatness modulus $\varpi(\tau, 0)$ is small it means that $\psi(\tau, e^{-y})$ is close to a constant multiple of $\Phi_{p,\gamma}(\tau, e^{-y})$, for y in any compact interval.

Theorem 4.11. For sufficiently large γ and sufficiently small $p > 0$ there exist positive constants δ_* and K_0 with the following property. If the scaled mean field and flatness modulus of h satisfy

$$\left|\frac{1}{\kappa(0)} - \frac{1}{\kappa_*}\right| + \varpi(0, 0) \leqslant \delta_*$$

where $\kappa_* = 1 + \frac{1}{p}$, then

$$\left|\frac{1}{\kappa(\tau)} - \frac{1}{\kappa_*}\right| + \varpi(\tau, 0) \leqslant K_0 \left(\left|\frac{1}{\kappa(0)} - \frac{1}{\kappa_*}\right| + \varpi(0, 0)\right)$$

for all $\tau \geqslant 0$.

If in addition $\psi_0(\cdot)$ is regularly varying with exponent p (equivalently, if $h(0, \cdot)$ is locally flat at ∞), then as $\tau \to \infty$ it holds $\kappa(\tau) \to \kappa_*$, $\varpi(\tau, 0) \to 0$ and $\lim_{\tau \to \infty} \psi(\tau, u) = \Psi_p(u)$ for all $u \in [0, 1]$.

Proof. We start by getting an a priori estimate on

$$\eta(\tau) := \frac{1}{x(\tau)} - \frac{1}{x_*} = \frac{1}{3}\left(\frac{1}{\kappa(\tau)} - \frac{1}{\kappa_*}\right),$$

where

$$x_* = \frac{3\kappa_*}{3\kappa_* - 1} = \frac{3 + 3/p}{2 + 3/p} \in \left[1, \frac{3}{2}\right].$$

Recall that $\Phi_{p,\gamma}(\tau, \cdot)$ is monotone increasing for all $\tau > 0$ if the parameter $\gamma > \gamma_0$.

Lemma 4.12. Assume $\gamma > \max(\gamma_0, 4)$, and assume that for some $\tau_* > 0$ we have $\varpi(\tau, 0) \leqslant \frac{1}{2}$ and $|\eta(\tau)| \leqslant \frac{1}{6}$ for $0 \leqslant \tau \leqslant \tau_*$. Then there exists $\bar{K} \geqslant 1$, independent of h, γ and p, such that

$$|\eta(\tau)| \leqslant e^{-\alpha(\tau)} \left(|\eta(0)| + \bar{K} \int_0^\tau e^{\alpha(s)} \alpha'(s)\, \varpi(s, 0)\, ds\right) \tag{42}$$

for all $0 \leqslant \tau \leqslant \tau_*$ with $\alpha(\tau) = \int_0^\tau \alpha'(s)\, ds$ and $\alpha'(\tau) \geqslant c e^{\tau/3}$ for some $c > 0$.

Proof. The function η is locally Lipschitz and is right and left differentiable, and from (26) we have

$$\partial_\tau^+\eta = -\frac{\partial_\tau^+ x}{x^2} = -\frac{\gamma e^{\tau/3}\psi(\tau,1)}{x(x-1)}\left(\frac{1}{x}-\int_0^1 \frac{\psi(\tau,u)}{\psi(\tau,1)}\frac{du}{3(1-u)^{2/3}}+\frac{e^{-\tau/3}}{3\gamma}\frac{x-1}{\psi(\tau,1)}\right). \tag{43}$$

With $\Phi = \Phi_{p,\gamma}$ as in (30) we obtain using (27) and (28) that

$$\frac{1}{x_*}-\int_0^1 \frac{\Phi(\tau,u)}{\Phi(\tau,1)}\frac{du}{3(1-u)^{2/3}}+\frac{e^{-\tau/3}}{3\gamma}\frac{x_*-1}{\Phi(\tau,1)}=0 \tag{44}$$

for all $\tau \geqslant 0$. Divide (43) by the quantity

$$\hat{\alpha}(\tau) := \gamma e^{\tau/3}\frac{\psi(\tau,1)}{x(\tau)(x(\tau)-1)}$$

and add (44). It follows that

$$\frac{\partial_\tau^+\eta}{\hat{\alpha}(\tau)} = -\eta(\tau)-I_1(\tau)+\frac{e^{-\tau/3}}{3\gamma}\left(\frac{x_*-1}{\Phi(\tau,1)}-\frac{x-1}{\psi(\tau,1)}\right) \tag{45}$$

where

$$I_1(\tau) := \int_0^1\left(\frac{\Phi(\tau,u)}{\Phi(\tau,1)}-\frac{\psi(\tau,u)}{\psi(\tau,1)}\right)\frac{du}{3(1-u)^{2/3}}.$$

In the last term of (45) we write, using $x_*-x = xx_*\eta$,

$$\frac{x_*-1}{\Phi(\tau,1)}-\frac{x-1}{\psi(\tau,1)} = \frac{xx_*\eta}{\Phi(\tau,1)}+(x-1)\left(\frac{1}{\Phi(\tau,1)}-\frac{1}{\psi(\tau,1)}\right).$$

To proceed we divide the mass conservation equation (23) by $\psi(\tau,1)$ and subtract the corresponding equation for Φ. This yields

$$\begin{aligned} I_2(\tau) &:= \int_0^1\left(\frac{\Phi(\tau,u)}{\Phi(\tau,1)}-\frac{\psi(\tau,u)}{\psi(\tau,1)}\right)du \\ &= \left(\frac{1}{\Phi(\tau,1)}-\frac{1}{\psi(\tau,1)}\right)\left(1-\frac{xe^{-\tau/3}}{\gamma}\right)-\frac{e^{-\tau/3}}{\gamma}\frac{xx_*\eta}{\Phi(\tau,1)}. \end{aligned} \tag{46}$$

Since $1 \leqslant x_* \leqslant \frac{3}{2}$, our hypothesis $|\eta(\tau)| < \frac{1}{6}$ implies $x(\tau) < 2$; hence for $\gamma > 4$ we have $1 - xe^{-\tau/3}/\gamma > \frac{1}{2}$, and we can use (46) to write (45) in the form

$$\partial_\tau^+ \eta = -\hat{\alpha}(\tau)((1-\delta_1(\tau))\,\eta(\tau) + I_1(\tau) + \tfrac{1}{3}\,\delta_2(\tau)\,I_2(\tau)) \tag{47}$$

where

$$\delta_2(\tau) = \frac{(x(\tau)-1)\,e^{-\tau/3}/\gamma}{1 - x(\tau)\,e^{-\tau/3}/\gamma} < 1$$

and

$$\delta_1(\tau) = \frac{e^{-\tau/3}}{3\gamma}\,\frac{xx_*}{\Phi(\tau, 1)}\,(1+\delta_2(\tau)) < \frac{4}{5}$$

since $\Phi(\tau, 1) > \int_0^1 \Phi(\tau, u)\,du = 1 - x_* e^{-\tau/3}/\gamma > \frac{5}{8}$.

We estimate $I_1(\tau)$ and $I_2(\tau)$ in (47) as follows. Using definition (41) we find

$$I_1(\tau) = \int_0^\infty (1 - e^{h(\tau,0)-h(\tau,y)})\,\frac{\Phi(\tau, e^{-y})}{\Phi(\tau, 1)}\,\frac{1}{3}\,(1-e^{-y})^{-2/3}\,e^{-y}\,dy. \tag{48}$$

We know that $\Phi(\tau, u) \leqslant \Phi(\tau, 1)$, and (39) yields

$$|h(\tau, 0) - h(\tau, y)| \leqslant (y+1)\,\varpi(\tau, 0).$$

In view of $|1-e^x| \leqslant |x|\,e^{|x|}$ for all x, and the presumption that $\varpi(\tau, 0) \leqslant \frac{1}{2}$, there exists $K_1 > 0$ such that

$$|I_1(\tau)| \leqslant \int_0^\infty \varpi(\tau, 0)(y+1)\,e^{\varpi(\tau,0)(y+1)}\,\tfrac{1}{3}\,(1-e^{-y})^{-2/3}\,e^{-y}\,dy \leqslant K_1 \varpi(\tau, 0). \tag{49}$$

In a similar way we also deduce the estimate

$$|I_2(\tau)| = \left|\int_0^\infty (1 - e^{h(\tau,0)-h(\tau,y)})\,\frac{\Phi(\tau, e^{-y})}{\Phi(\tau, 1)}\,e^{-y}\,dy\right| \leqslant 3K_1 \varpi(\tau, 0). \tag{50}$$

With

$$\alpha(\tau) := \int_0^\tau \hat{\alpha}(s)(1-\delta_1(s))\,ds$$

we find using (47), (49) and (50) that

$$|\partial_\tau^+ \eta(\tau) + \alpha'(\tau)\, \eta(\tau)| \leqslant \alpha'(\tau)\, \bar{K} \varpi(\tau, 0)$$

where $\bar{K} = 10\, K_1$. The inequality (42) follows in standard fashion.

It remains to show that

$$\alpha'(\tau) \geqslant \frac{e^{\tau/3}}{5} \frac{\psi(\tau, 1)}{x(\tau)(x(\tau)-1)} \geqslant c e^{\tau/3} \tag{51}$$

for some $c > 0$. As long as $|\eta(\tau)| < \frac{1}{6}$ we have $x(\tau) < 2$ and hence $\psi(\tau, 1) > \int_0^1 \psi(\tau, u)\, du = 1 - x e^{-\tau/3}/\gamma > \frac{1}{2}$. Furthermore

$$\frac{1}{x} \geqslant \frac{1}{x_*} - \frac{1}{6} \geqslant \frac{1}{2}$$

which implies (51) and finishes the proof of the lemma. ∎

To proceed further we have to estimate the flatness modulus in terms of η, by following characteristics backwards. The estimate differs from that in ref. 14 and we have to redo it.

Lemma 4.13. For γ sufficiently large and $0 < p \leqslant 1$, there is a constant C and a positive decreasing function $G_{p,\gamma} : [0, \infty] \to \mathbb{R}$ with $\int_0^\infty G_{p,\gamma}(\tau)\, d\tau \leqslant C(p + \gamma^{-1})$ such that the following holds. Assume that for some $\tau_* > 0$ we have

$$|\eta(\tau)| = \frac{1}{3} \left| \frac{1}{\kappa(\tau)} - \frac{1}{\kappa_*} \right| \leqslant \frac{p}{12(p+1)^2}$$

for $0 \leqslant \tau \leqslant \tau_*$. Then

$$\varpi(\tau, 0) \leqslant \varpi(0, \tau/2p) + \int_0^\tau G_{p,\gamma}(\tau - s)\, |\eta(s)|\, ds \tag{52}$$

for all $0 \leqslant \tau \leqslant \tau_*$.

Proof. Given $\tilde{\tau}, \tilde{y} > 0$, let $\mathcal{Y}(\tilde{\tau}, \tilde{y})$ denote the characteristic satisfying

$$\frac{\partial \mathcal{Y}}{\partial \tau} = -(\kappa(\tau)\, Q(e^{-\mathcal{Y}}) - 1), \qquad \mathcal{Y}(\tilde{\tau}, \tilde{y}) = \tilde{y}. \tag{53}$$

As long as $|\kappa^{-1}-\kappa_*^{-1}| \leqslant p/4(p+1)^2$ we have $\kappa \leqslant 2\kappa_*$ and $|\kappa-\kappa_*| \leqslant 1/2p$. Then for $0 \leqslant \tau \leqslant \tilde{\tau}$ we have $-\partial\mathscr{Y}/\partial\tau \geqslant \kappa(\tau)-1 \geqslant 1/2p$, hence

$$\mathscr{Y}(\tau, \tilde{y}) \geqslant \tilde{y}+(\tilde{\tau}-\tau)/2p. \tag{54}$$

Suppose now that $0 \leqslant \tilde{y} \leqslant \tilde{y}_1 \leqslant \tilde{y}_2 \leqslant \tilde{y}+1$. Let

$$\begin{aligned} \mathscr{Y}_1(\tau) &= \mathscr{Y}(\tau, \tilde{y}_1), \qquad & y_1 &= \mathscr{Y}_1(0), \\ \mathscr{Y}_2(\tau) &= \mathscr{Y}(\tau, \tilde{y}_2), \qquad & y_2 &= \mathscr{Y}_2(0). \end{aligned} \tag{55}$$

Since $y \mapsto -Q(e^{-y})$ is increasing, it follows that $\mathscr{Y}_2(\tau)-\mathscr{Y}_1(\tau)$ is increasing in τ (i.e., characteristics are diverging); hence $\mathscr{Y}_2(\tau)-\mathscr{Y}_1(\tau) \leqslant 1$ for $0 \leqslant \tau \leqslant \tilde{\tau}$.

Next we study how pairs of values of $h(\tau, y)$ vary along characteristics. Using (19), and (24) with $\kappa = \kappa_*$ and $\psi = \Phi_{p,\gamma}$, we find

$$h(\tilde{\tau}, \tilde{y}_j)-h(0, y_j) = \int_0^{\tilde{\tau}} (\kappa(\tau)-\kappa_*)\, F(\tau, e^{-\mathscr{Y}_j(\tau)})\, d\tau \tag{56}$$

where we compute that

$$F(\tau, u) = Q(u)\, u\, \frac{\partial_u \Phi_{p,\gamma}(\tau, u)}{\Phi_{p,\gamma}(\tau, u)} = \frac{Q(u)}{\kappa_* Q(u)-1} \left(1-\frac{1}{3}\frac{g(\tau, u)}{1-g(\tau, u)}\right) \tag{57}$$

with

$$g(\tau, u) = \frac{e^{-\tau/3}}{\gamma}\, c_p \Psi_p(u)^{1/3}. \tag{58}$$

Let $F_j(\tau) = F(\tau, e^{-\mathscr{Y}_j(\tau)})$ and define $Q_j(\tau)$ and $g_j(\tau)$ similarly. Then

$$\begin{aligned} H(\tilde{\tau}, \tilde{y}_1, \tilde{y}_2) &:= (h(\tilde{\tau}, \tilde{y}_2)-h(\tilde{\tau}, \tilde{y}_1))-(h(0, y_2)-h(0, y_1)) \\ &= \int_0^{\tilde{\tau}} (\kappa(\tau)-\kappa_*)(F_2(\tau)-F_1(\tau))\, d\tau. \end{aligned} \tag{59}$$

To estimate $|F_2(\tau)-F_1(\tau)|$ we use that $a_2 b_2-a_1 b_1 = (a_2-a_1)\, b_2 + a_1(b_2-b_1)$ and note that $\kappa_* Q(u)-1 \geqslant \kappa_*-1 = 1/p$ and $g(\tau, u) \leqslant \frac{1}{2}$ for $\gamma > \gamma_0$. Then

$$\left|\frac{Q_2(\tau)}{\kappa_* Q_2(\tau)-1}-\frac{Q_1(\tau)}{\kappa_* Q_1(\tau)-1}\right| \leqslant p^2\, |Q_1(\tau)-Q_2(\tau)| \leqslant 2p^2 e^{-\mathscr{Y}_1(\tau)} \tag{60}$$

since $0 \leqslant u_2 \leqslant u_1 \leqslant 1$ implies $0 \leqslant Q(u_1) - Q(u_2) \leqslant Q(u_1) - Q(0) \leqslant 2u_1$. Also,

$$\frac{1}{3}\left|\frac{g_2(\tau)}{1-g_2(\tau)} - \frac{g_1(\tau)}{1-g_1(\tau)}\right| \leqslant \frac{4}{3}\,|g_2(\tau) - g_1(\tau)| = \frac{4}{3}\,g_1(\tau)\left(1 - \frac{g_2(\tau)}{g_1(\tau)}\right). \tag{61}$$

From (27) we see that

$$u\,\frac{\partial}{\partial u}\ln g(\tau, u) = \frac{1}{3}\,\frac{1}{\kappa_* Q(u) - 1} \leqslant \frac{p}{3}, \tag{62}$$

whence $g_2(\tau)/g_1(\tau) \geqslant e^{-p/3}$ since $\mathscr{Y}_2(\tau) - \mathscr{Y}_1(\tau) \leqslant 1$.

Collecting these estimates we find with $Q(u)/(\kappa_* Q(u) - 1) \leqslant p$ that

$$|F_2(\tau) - F_1(\tau)| \leqslant 2p^2 e^{-\mathscr{Y}_1(\tau)} + \tfrac{4}{3}\,p(1 - e^{-p/3})\,g_1(\tau).$$

Due to the estimates on $\Psi_p(u)$ in (29) and the normalization (31) that determines c_p, we have $c_p \Psi_p(u)^{1/3} \leqslant \hat{C} u^{p/3}$ for $0 < p \leqslant 1$ for some $\hat{C}$ independent of p. We then deduce by using (54) that

$$\begin{aligned} |F_2(\tau) - F_1(\tau)| &\leqslant 2p^2 e^{-\mathscr{Y}_1(\tau)} + \frac{4\hat{C}}{9\gamma}\,p^2 e^{-\mathscr{Y}_1(\tau)\,p/3} \\ &\leqslant p^2 (2e^{-(\tilde{\tau}-\tau)/2p} + \hat{C}\gamma^{-1} e^{-(\tilde{\tau}-\tau)/6}). \end{aligned} \tag{63}$$

Since $|\kappa - \kappa_*| = 3\kappa\kappa_*\,|\eta| \leqslant 6(1 + 1/p)^2\,|\eta|$ we obtain

$$|H(\tilde{\tau}, \tilde{y}_1, \tilde{y}_2)| \leqslant \int_0^{\tilde{\tau}} G_{p,\gamma}(\tilde{\tau} - \tau)\,|\eta(\tau)|\,d\tau \tag{64}$$

with

$$G_{p,\gamma}(s) = 12(p+1)^2\,(e^{-s/2p} + \hat{C}\gamma^{-1} e^{-s/6}). \tag{65}$$

Since $\tilde{y}_1, \tilde{y}_2 \in [\tilde{y}, \tilde{y}+1]$ are arbitrary and $\tilde{y} + \tilde{\tau}/2p \leqslant y_1 \leqslant y_2 \leqslant y_1 + 1$, estimate (64) directly implies (52). This finishes the proof of the Lemma. ▮

We now proceed with the proof of Theorem 4.11. Let $\hat{\eta}(\tau) := \sup_{0 \leqslant s \leqslant \tau} |\eta(s)|$. Using (52) in (42) and using $\bar{K} \int_0^\infty G_{p,\gamma}(\tau)\,d\tau \leqslant C(p + \gamma^{-1}) < \frac{1}{2}$ for sufficiently large γ and small $p > 0$, we find that as long as $|\eta(\tau)| \leqslant p/12(p+1)^2 < \frac{1}{6}$, we have

$$\hat{\eta}(\tau) \leqslant 2(|\eta(0)| + \bar{K}\varpi(0, 0)) \leqslant 2(\tfrac{1}{3} + \bar{K})\,\delta_*. \tag{66}$$

Thus we may take $\delta_* = p/24(\frac{1}{3}+\bar{K})(p+1)^2$ to find $|\eta(\tau)| \leqslant p/12(p+1)^2$ for all $\tau > 0$. Plugging (66) into (52) gives for sufficiently large γ and small $p > 0$ that $\varpi(\tau, 0) \leqslant 2\varpi(0, 0) + |\eta(0)|$ which proves the bound asserted in the theorem with $K_0 = 2 + 6\bar{K}$.

Now assume that $h(0, \cdot)$ is locally flat at ∞ which implies $\varpi(0, s/2p) \to 0$ as $s \to \infty$. We get from (42) and (52), with

$$\zeta(\tau) := e^{-\alpha(\tau)} \left(|\eta(0)| + \bar{K} \int_0^\tau e^{\alpha(s)} \alpha'(s) \varpi(0, s/2p)\, ds \right),$$

that

$$|\eta(\tau)| \leqslant \zeta(\tau) + \bar{K} e^{-\alpha(\tau)} \int_0^\tau e^{\alpha(s)} \alpha'(s) \int_0^s G_{p,\gamma}(s-\hat{s})\, |\eta(\hat{s})|\, d\hat{s}\, ds.$$

We know that $\zeta(\tau) \to 0$ as $\tau \to \infty$, that $|\eta(\tau)| \leqslant \hat{K} = K_0(\varpi(0, 0) + |\eta(0)|)$ for all $\tau \geqslant 0$ and hence that $\eta_\infty := \limsup_{\tau \to \infty} |\eta(\tau)| < \infty$. Given $\varepsilon > 0$, we choose a time τ_0 such that $|\eta(\tau)| \leqslant \eta_\infty + \varepsilon$ for all $\tau \geqslant \tau_0$. Then using $\bar{K} \int_0^\infty G_{p,\gamma}(\tau)\, d\tau < \frac{1}{2}$ we get

$$|\eta(\tau)| \leqslant \zeta(\tau) + \hat{K} e^{-\alpha(\tau)} \int_0^{\tau_0} e^{\alpha(s)} \alpha'(s) \int_0^s G_{p,\gamma}(s-\hat{s})\, d\hat{s}\, ds + \tfrac{1}{2}(\eta_\infty + \varepsilon),$$

and taking $\tau \to \infty$ it follows $\eta_\infty \leqslant \frac{1}{2}(\eta_\infty + \varepsilon)$ which implies $\eta_\infty = 0$.

The convergence of $\varpi(\tau, 0)$ now follows from (52) and the convergence of ψ from Theorem 4.8. ∎

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REFERENCES

1. N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Encycl. Math. Appl. v. 27 (Cambridge Univ. Press, Cambridge, 1987).
2. J. Carr and O. Penrose, Asymptotic behaviour in a simplified Lifshitz-Slyozov equation, *Physica D* **124**:166–176 (1998).
3. J.-F. Collet and T. Goudon, On solutions of the Lifshitz-Slyozov model, *Nonlinearity* **13**:1239–1262 (2000).
4. H. Federer, *Geometric Measure Theory* (Spinger-Verlag, New York, 1969).
5. A. Friedman and B. Ou, A model of crystal precipitation, *J. Math. Anal. Appl.* **137**:550–575 (1989).

6. A. Friedman, B. Ou, and D. Ross, Crystal Precipitation with Discrete Initial Data, *J. Math. Anal. Appl.* **137**:576–590 (1989).
7. B. Giron, B. Meerson, and P. V. Sasorov, Weak selection and stability of localized distributions in Ostwald ripening, *Phys. Rev. E* **58**:4213–4217 (1998).
8. S. C. Hardy and P. W. Voorhees, Ostwald Ripening in a system with a high volume fraction of coarsening phase, *Metallurgical Trans. A* **19**:2713–2721 (1988).
9. Ph. Laurencot, On solutions to the Lifshitz-Slyozov model, *Ind. Univ. Math. J.*, to appear.
10. I. M. Lifshitz and V. V. Slyozov, The kinetics of precipitation from supersaturated solid solutions, *J. Phys. Chem. Solids* **19**:35–50 (1961).
11. B. Meerson and P. V. Sasorov, Domain stability, competition, growth and selection in globally constrained bistable systems, *Phys. Rev. E* **53**:3491–4 (1996).
12. B. Niethammer, Derivation of the LSW theory for Ostwald ripening by homogenization methods, *Arch. Rat. Mech. Anal.* **147**:119–178 (1999).
13. B. Niethammer and R. L. Pego, On the initial value problem in the Lifshitz–Slyozov–Wagner theory of Ostwald ripening, *SIAM J. Math. Anal.* **31**:467–485 (2000).
14. B. Niethammer and R. L. Pego, Non-self-similar behavior in the LSW theory of Ostwald Ripening, *J. Stat. Phys.* **95**:867–902 (1999).
15. O. Penrose, The Becker–Döring equations at large times and their connection with the LSW theory of coarsening, *J. Stat. Phys.* **89**:305–320 (1997).
16. E. Seneta, *Regularly Varying Functions*, Lec. Notes in Math. Vol. 508 (Springer-Verlag, New York, 1976).
17. J. J. L. Velázquez, The Becker–Döring equations and the Lifshitz-Slyozov theory of coarsening, *J. Stat. Phys.* **92**:195–236 (1998).
18. P. W. Voorhees, The theory of Ostwald ripening, *J. Stat. Phys.* **38**:231–252 (1985).
19. P. W. Voorhees, Ostwald ripening of two-phase mixtures, *Ann. Rev. Mater. Sci.* **22**:197–215 (1992).
20. C. Wagner, Theorie der Alterung von Niederschlägen durch Umlösen, *Z. Elektrochem.* **65**:581–594 (1961).